Stationary Values

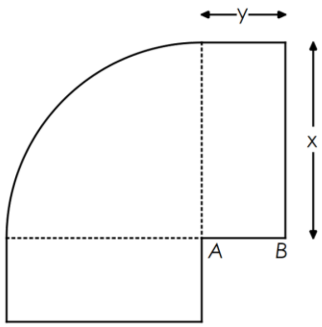

Figure 1

Figure 1 shows the plan view of the design for a swimming pool.

The pool is modelled as a quarter circle joined to two equal sized rectangles as shown.

Given that

- the quarter circle has radius x metres
- the rectangles each have length x metres and width *y* metres
- the total surface area of the swimming pool is 169 m^2

a. show that, according to the model, the perimeter *P* metres of the swimming pool is given by

$$P = 2x + \frac{338}{x}$$

(5 marks)

b. Use calculus to find the value of x for which *P* has a stationary value.

(4 marks)

c. Prove, by further calculus, that the value of x gives a minimum value of *P*.

(2 marks)

Access to the pool is by side *AB* shown in Figure 1.

Given that AB must be at least 1.3 m,

d. determine, according to the model, whether the swimming pool with the minimum perimeter would be suitable.

(2 marks)

a. The surface area of the pool is made of two identical rectangles and a quarter circle:

$$\text{Surface area} = 2xy + \frac{\pi x^2}{4}$$

1 mark

Equate this to 169 to get an expression for *y*:

$$2xy + \frac{\pi x^2}{4} = 169 \Rightarrow 2xy = 169 - \frac{\pi x^2}{4}$$

$$y = \frac{169}{2x} - \frac{\pi x}{8}$$

1 mark

The perimeter of the pool is made $2x$, $4y$ and a quarter of the circumference of the circle.

$$P = 2x + 4y + \frac{2\pi x}{4}$$

1 mark

$$P = 2x + 4\left(\frac{169}{2x} - \frac{\pi x}{8}\right) + \frac{2\pi x}{4}$$

1 mark

$$P = 2x + \frac{338}{x}$$

1 mark

b. Differentiate the expression for *P* and equate to 0 to find the value of *x* for a stationary point:

1 mark

$$\frac{dP}{dx} = 2 - 338x^{-2}$$

1 mark

$$2 - 338x^{-2} = 0$$

$$x^2 = 169$$

1 mark

$$x = 13$$

1 mark

c. Use the second derivative to show that the stationary point gives a minimum.

$$\frac{d^2P}{dx^2} = 676x^{-3}$$

1 mark

$676 \times 13^{-3} = 0.30... > 0$ Hence a minimum

1 mark

d. Substitute $x = 13$ into the equation for *y*:

$$y = \frac{676 - \pi \times 13^2}{8 \times 13}$$

1 mark

$y = 1.39$ m, so the minimum perimeter would be suitable.

1 mark