Differentiation

A company makes drinks containers out of metal.

The containers are modelled as closed cylinders with base radius r cm and height h cm and the capacity of each container is 345 cm³

The metal used

- for the circular base and the curved sides costs 0.05 pence/cm²

- for the circular top costs 0.08 pence/cm²

Both metals used are of negligible thickness.

a. Show that the total cost, C pence, of the metal for one container is given by

$$C = 0.13\pi r^2 + \frac{34.5}{r}$$

(4 marks)

b. Use calculus to find the value of r for which C is a minimum, giving your answer to 3 significant figures.

(4 marks)

c. Using $\dfrac{d^2C}{dr^2}$ prove that the cost is minimised for the value of r found in part b.

(2 marks)

d. Hence find the minimum value of C, giving your answer to the nearest integer.

(2 marks)

a. Begin with the volume of the cylinder and rearrange to get an expression for h

$$V = \pi r^2 h \text{ so } \pi r^2 h = 345 \Rightarrow h = \frac{345}{\pi r^2}$$

1 mark

The cost, C, of the container is related it's the surface area.

$$C = 0.05(\pi r^2 + 2\pi rh) + 0.08\pi r^2$$

1 mark

the material for the lid has a different price per cm²

Substitute the expression for h and simplify.

$$C = 0.05\pi r^2 + 0.1\pi r \times \frac{345}{\pi r^2} + 0.08\pi r^2$$

1 mark

$$C = 0.13\pi r^2 + \frac{34.5}{r}$$

1 mark

b. To find the minimum, differentiate the expression for C

$$C = 0.13\pi r^2 + 34.5r^{-1} \Rightarrow \frac{dC}{dr} = 0.26\pi r - 34.5r^{-2}$$

2 marks

Put this equal to zero and solve for r.

$$0.26\pi r - 34.5r^{-2} = 0$$

$$0.26\pi r = \frac{34.5}{r^2}$$

$$r^3 = \frac{34.5}{0.26\pi}$$

1 mark

$$r = \sqrt[3]{\frac{34.5}{0.26\pi}} = 3.48\ldots$$

1 mark

c. To check that this gives the minimum cost, use the second derivative.

$$\frac{d^2C}{dr^2} = 0.26\pi + 69r^{-3}$$

$$= 0.26\pi + \frac{69}{3.48^3}$$

1 mark

$$\frac{d^2C}{dr^2} = 2.45\ldots > 0 \text{ so this gives minimum cost}$$

1 mark

d. Substitute the value of r into the expression for C to get the cost

$$C = 0.13\pi \times 3.48^2 + \frac{34.5}{3.48}$$

1 mark

$$C = 15$$

1 mark